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CYCLES AND PATHS IN BIPARTITE TOURNAMENTS WITH SPANNING CONFIGURATIONS

R. HÄGGKVIST and Y. MANOUSSAKIS

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We give necessary and sufficient conditions in terms of connectivity and factors for the existence of hamiltonian cycles and hamiltonian paths and also give sufficient conditions in terms of connectivity for the existence of cycles through any two vertices in bipartite tournaments.

1. Introduction and Terminology

By Camion's well known theorem [5], any ordinary tournament is hamiltonian if and only if it is strong. Unfortunately this result does not have a direct analogue in bipartite tournaments since there are infinite families of strong bipartite tournaments without hamiltonian cycles, or even without cycles of length more than four, and strong connectivity, even if high, does not seem to play an important role for the existence of large cycles in bipartite tournaments. More precisely, as pointed out in [7, 1], there are k-strongly connected nonhamiltonian bipartite tournaments on $4k+2$ vertices and there are strong bipartite tournaments with pairs of vertices which are not on a common cycle. So it is natural to look for additional conditions which would imply the existence of large cycles and paths in bipartite tournaments. Here we shall consider as additional conditions the existence of spanning diregular subgraphs.

In particular, in the second section we give necessary and sufficient conditions in terms of connectivity and factors for classes of bipartite tournaments to be hamiltonian. As an immediate consequence, we obtain sufficient conditions for the existence of cycles of many lengths through any given vertex of a bipartite tournament. We also give necessary and sufficient conditions in terms of factors for the existence of hamiltonian paths in bipartite tournaments. Next, we conjecture that in k-strongly connected bipartite tournament any k vertices are on a common cycle and prove this for $k=2$.

Formally, throughout this paper $B=(X, Y, E)$ denotes a bipartite tournament of order n with partition (X, Y), where $|X|=a\leqq b=|Y|$ and $n=a+b$. Then $V(B)(=X\cup Y)$ denotes the set of vertices and $E(B)$ denotes the set of arcs of B. When $a=b$, we say that B is balanced. If x and y are vertices of B, then we shall say that x dominates y if the arc (x, y) is present. If x is a vertex of B and C a subset of $V(B)$, then we define $\Gamma_C^-(x)$ and $\Gamma_C^+(x)$ to be the set of vertices of C which, respectively, dominate, and are dominated by, the vertex x. The out-

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degree, indegree and degree of a vertex x in C are defined as $|\Gamma_C^+(x)|$, $|\Gamma_C^-(x)|$ and $|\Gamma_C^-(x)|+|\Gamma_C^+(x)|$ respectively and are denoted $d_C^+(x)$, $d_C^-(x)$ and $d_C(x)$ respectively. When $V(C)=V(B)$ we write $d^+(x)$ (resp $d^-(x), d(x)$) instead of $d_C^+(x)$ (resp $d_C^-(x), d_C(x)$). By $G[X, Y]$ we denote the induced subgraph of G consisting of all edges in G with one end in X and the other in Y, where G is an undirected graph and X, Y are subsets of $V(G)$. $G[X]$ is the graph $G[X, X]$. A digraph D is said to be k-cyclic if any k vertices of D are contained in a common cycle.

A matching $M_{X,Y}$ from X into Y is a set of arcs of B with origin in X and terminus in Y such that no two arcs in $M_{X,Y}$ are adjacent. The bipartite tournament B is said to be r-diregular if and only if $d^+(x)=r=d^-(x)$, for all $x\in V(B)$. An (1, 1)-factor, or more simply, a factor F is a spanning 1-diregular subgraph of B. A factor F consists of the disjoint union of cycles $C_1, C_2, \ldots, C_m$, where $m\geq 1$, and clearly a bipartite tournament admits a factor F if and only if $|X|=|Y|$ and moreover it has a matching $M_{X,Y}$ and a matching $M_{Y,X}$ each of cardinality $|X|$.

In the proof of Theorem 2.1 below we shall use the following theorem by M. Bankfalvi and Zs. Bankfalvi [2].

Theorem A (M. Bánkfalvi and Zs. Bánkfalvi). *Let G be a complete graph on $2n$ vertices with edges coloured red and blue. Then G has an alternating hamiltonian cycle unless either* (i) *has no alternating 2-factor (i.e. the union of red and blue 1-factors) or* (ii) *the vertex set of G can be partitioned into nonempty sets R, S, T with $|R|=|S|\geq 2$ such that all RR and RT-edges are red while all SS and ST-edges are blue.*

This is not the formulation given in [2], but they derive this structural information on pages 12 and 13 in their paper.

In the next section we shall occassionally refer to the bipartite tournament $B(|K|, |L|, |P|, |N|)$ which first appeared in [7] under a different name. Its definition is: Let K, L, P and N be independent sets. The graph $B(|K|, |L|, |P|, |N|)$ consists from the disjoint union of K, L, P, N and all the arcs from K to L, from L to P, from P to N and from N to K. Some interesting properties of this bipartite tournament are the following: (1) It is $\min(|K|, |L|, |P|, |N|)$-strongly connected (2) It is hamiltonian but not bipancyclic when $|K|=|L|=|P|=|N|$ and (3) It is not 2-cyclic when $\min(|K|, |L|, |P|, |N|)=1$ and $\max(|K|, |L|, |P|, |N|)>1$.

2. Cycles in bipartite tournaments

In Theorem 2.1 we shall give necessary and sufficient conditions for bipartite tournaments to be hamiltonian.

Theorem 2.1. *Any bipartite tournament B is hamiltonian if and only if both* (i) *and* (ii) *below hold:*
(i) *B is strong and*
(ii) *B has a factor F.*

Proof. Assume that B has a bipartition (X, Y), and that B fails to contain a hamiltonian cycle (otherwise there is nothing to prove). We shall show that either B fails to contain a factor or else is not strong. To do so we consider a bicoloured complete graph G on $X\cup Y$ obtained from B as follows: every edge of the form XX is red, every YY-edge is blue and an XY-edge xy (x in X and y in Y) is blue in G if and only if (x, y) is an arc in B. Otherwise xy is red. Now note that every alternating

2-factor in G is also alternating 2-factor in $G[X, Y]$ and corresponds to a factor in B. The same goes for hamiltonian cycles. Consequently we know that G has no alternating hamiltonian cycle. It follows from Theorem A that G either fails to contain an alternating 2-factor in which case B has no factor or else $V(G)$ can be partitioned into three non empty sets R, S, T with $|R|=|S|\geqq 2$ such that $G[R]$ and $G[R, T]$ have red edges only, while $G[S]$ and $G[S, T]$ are blue. The fact that $G[R]$ has red edges only implies that $R'=R\cap X$ is non empty, and similarly we get that $S'=S\cap Y$ must be non empty. The fact that every every edge in $G[R', T\cup(R-R')]$ is red means that every edge in B with one end in R' and the other in $(T\cup(R-R'))\cap Y$ must be directed into R', and likewise every edge in B with one end in S' and the other in $(T\cup(S-S')\cap X)$ must be directed into S'. Consequently B cannot be strong (note that $R'\cup S'$, $((T\cup(R-R'))\cap Y)\cup$ $\cup((T\cup(S-S'))\cap X)$ gives a directed cut in B). The proof of the theorem is complete. ∎

Remark. Theorem 2.1 can be extended to semi-complete bipartite tournaments using a different, longer, proof (for more details see [8]).

From Theorem 2.1 we obtain the following corollaries.

Corollary 2.2. *The problem to test whether a bipartite tournament contains a hamiltonian cycle is solvable in $O(n^{2.5})$ time.*

Proof. There are $O(n^2)$ algorithms to test whether a bipartite tournament is strong and there are $O(n^{2.5})$ algorithms to test whether a bipartite tournament contains a (1, 1)-factor (see [10]). Consequently the conclusion follows immediately from Theorem 2.1. ∎

In Corollary 2.4 we shall give necessary and sufficient conditions for the existence of hamiltonian paths in bipartite tournaments. For that we need the following fundamental lemma which gives some information about the structure of bipartite tournaments containing a factor.

Lemma 2.3. *Let B a bipartite tournament which admits a factor. Then B is not strong if and only if there exists a factor F in B consisting of cycles $C_1, C_2, \ldots, C_m$, $m\geqq 2$, such that there are arcs from a cycle C_i to a cycle C_j if and only if $i<j$.*

Proof. To prove sufficiency it suffices to study the structure of B. To prove necessity it suffices to take a factor F with cycles $C_1, \ldots, C_m$ such that m is the minimum possible and then by using Theorem 2.1 verify that F satisfies the conclusion of the lemma. ∎

Corollary 2.4. *Any bipartite tournament $B=(X, Y, E)$ has a hamiltonian path if and only if one of the conditions below holds.*

(i) *B has a factor*

(ii) *B has an "almost factor" that is to say a spanning subgraph S which differs from a factor by the direction of exactly one arc.*

(iii) *$|Y|-|X|=1$ and moreover B has a matching $M_{X,Y}$ and a matching $M_{Y,X}$ each of cardinality $|X|$.*

Proof. The necessity is obvious so assume that B is as described in the theorem. If (i) or (ii) holds, then B has an arc (x, y) such that $B'=(B-(x, y))\cup\{(y, x)\}$

has a factor. By Lemma 2.3, $B'-(y, x)=B-(x, y)$ has a hamiltonian path. Assume finally that (iii) holds. Let y_1 (resp. y_2) be the vertex of Y which is not saturated by $M_{X,Y}$ (resp. by $M_{Y,X}$). Now, add a new vertex x in X and all arcs incident with x in such a way that x dominates y_1 and is dominated by y_2 and such that the resulting bipartite tournament B' has as few strong components as possible. Clearly, B' has a factor. The minimality property of B' implies that B' is strong. By Theorem 2.1, B' is hamiltonian and therefore there exists a hamiltonian path in B, as required. The proof of the corollary is complete. ▮

Next proposition generalises a theorem proved by L. W. Beineke and C. H. C. Little in [3] and can clearly be used to extend Theorem 2.1.

Proposition 2.5. *Let $B=(X, Y, E)$ be a hamiltonian bipartite tournament and let x be any vertex of B. If k is any even integer, $4\leqq k\leqq n$, then either B contains a cycle of length k through x or else $n=4r$ and B is isomorphic to the bipartite tournament $B(r, r, r, r)$ defined at the end of the introduction.*

Proof. Induction over n. Assume $n\geqq 8$ since it is easy to see that the proposition is true for smaller values of n. Let now x be a vertex of B and assume that B is not isomorphic to $B(r, r, r, r)$. Since B is hamiltonian it suffices to prove that there exist a cycle of each even length k, $4\leqq k\leqq n-2$, through x. Moreover since B is bipancyclic (see [3]) it has a cycle C of length $n-2$. Put C: $x_1\to y_1\to \dots\to x_{\frac{n-2}{2}}\to$ $\to y_{\frac{n-2}{2}}\to x_1$ and let B' denote the subgraph induced by $V(C)$.

Assume first that there exists a cycle C of length $n-2$ through x. Then by applying the induction hypothesis on B' we conclude that either there exists a cycle of each even length k, $4\leqq k\leqq n-2$, through x in B', i.e. in B, as required, or else B' is isomorphic to $B(p, p, p, p)$, where $n=4p+2$. In this last case, by studing the structure of B, we can easily see that the conclusion of the proposition is true.

Assume next that there is no cycle of length $n-2$ through x. It follows that x is in $B-C$ and then we can easily deduce that either $d^+{}_C(x)=0$ or $d^-{}_C(x)=0$ holds for otherwise, by using standard arguments, one could find a cycle of length $n-2$ through x contrary to our assumption. Now let y be the second vertex of $B-C$ and assume w.l.of.g. that x dominates y and that $d^+{}_C(x)=0$ holds. Since B is strong it follows that $d^-{}_C(x)=\frac{|C|}{2}$ and that $d^+{}_C(y)=0$. Now take a vertex x_i of C such that the arc (y, x_i) is present. Then the arc $\left(y_{i+\frac{k}{2}-2}, x\right)$ is also present and therefore the cycle $x\to y\to x_i\to \dots\to y_{i+\frac{k}{2}-2}\to x$ is of length k through x. This completes the proof of the proposition. ▮

Perhaps Theorem 2.1 can be extended as follows:

Conjecture 2.6. *Let $D=(X, Y, E)$ be a strong bipartite digraph with bipartition (X, Y). Assume furthermore that D has a factor and that for any vertex v of D we have $d(v)>|X|$. Then D is hamiltonian unless it is isomorphic to the digraph with vertex set $X\cup Y$, where $X=\{x_1, x_2, x_3\}$, $Y=\{y_1, y_2, y_3\}$ and arc set $\{(x_1, y_1), (y_1, x_2), (x_2, y_2), (y_2, x_1)\}\cup\{(x_3, y_i), (y_i, x_3), (y_3, x_i), (x_i, y_3), 1\leqq i\leqq 2\}$.*

Moreover D is bipancyclic if it is hamiltonian and different from a symmetric cycle of length 6.

Note that, if true, the above conjecture would be best possible. In fact, consider the bipartite digraph obtained from the disjoint union of two copies of $K^*_{\frac{a}{2},\frac{a}{2}}$ on which we suppose that exactly one vertex of the first copy dominates and is dominated by at least one (but not necessarily the same) vertex of the same partition set in the second copy. Clearly the obtained digraph is not hamiltonian.

We conclude this section with a result on bipartite tournaments where we make no assumptions about factors. Recall the following unsolved problem of J. C. Bermond and L. Lovász [4].

Problem (Bermond and Lovász). *Does there exist a natural number k such that for every k-strongly connected digraph each pair of vertices belongs to a common cycle? ($k \geqq 6$ for digraphs.)*

In the next theorem we shall prove that $k=2$ suffices in the above problem if we replace "digraph" by "bipartite tournament".

Theorem 2.7. *Any 2-strongly connected bipartite tournament is 2-cyclic.*

Proof. Let x and y be two vertices of B and then consider two pairwise vertex-disjoint paths P_1 and P_2 from x to y. If one of them has length 2, then delete the intermediate vertex and consider a path from y to x. If, on the other hand, they both have length at least four, then assume that they are the smallest possible. Then the bipartite tournament induced by P_2 has a path P_3 from y to x and now $P_1 \cup P_3$ is a cycle through x and y. The proof of the theorem is complete. ∎ (The current proof of Theorem 2.7 was suggested by C. Thomassen.)

The above theorem is best possible because of the strong bipartite tournament $B(|Y|-2, |X|-1, 2, 1)$, where $|Y| > |X| \geqq 2$. However, $B(|Y|-2, |X|-1, 2, 1)$ is not the only strong non-2-cyclic bipartite tournament. Consider, for example, $2k$ vertices $x_1, y_1, x_2, y_2, \ldots, x_k, y_k$ and add the arcs (x_i, y_j) if and only if $i \leqq j$ and the arcs (y_i, x_j) if and only if $i < j$. Next, reverse the arc (x_1, y_k), add a new vertex x and then add the arcs (y_j, x), $1 \leqq j \leqq i-1$ and the arcs (x, y_j), $1 \leqq j \leqq k$, for i arbitrarily chosen such that $2 \leqq i \leqq k-2$. Clearly the resulting bipartite tournament is strong, however the vertices x and x_i are not on a common cycle.

Theorem 2.7 and the $(k-1)$-strongly connected bipartite tournament $B(|Y|-k, |X|-k+1, k, k-1)$, $|Y| \geqq |X| \geqq 2k-2 \geqq 2$ show that the following conjecture, if true, could be the best possible.

Conjecture 2.8. *Any k-strongly connected bipartite tournament is k-cyclic.*

Note that the 2-cyclic problem is NP-complete for digraphs [6]. However in [9] Z. Tuza and the second author obtained a polynomial algorithm which finds a cycle, if there exists, through any two vertices in a bipartite tournament. In the same reference, they give another polynomial algorithm for finding hamiltonian cycles in bipartite tournaments.

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R. Häggkvist

Dept. of Mathematics
University of Stockholm
Box 6701 S—11385
Stockholm, Sweden

Y. Manoussakis

Université PARIS—XI, LRI bât 490
91405 Orsay Cedex France